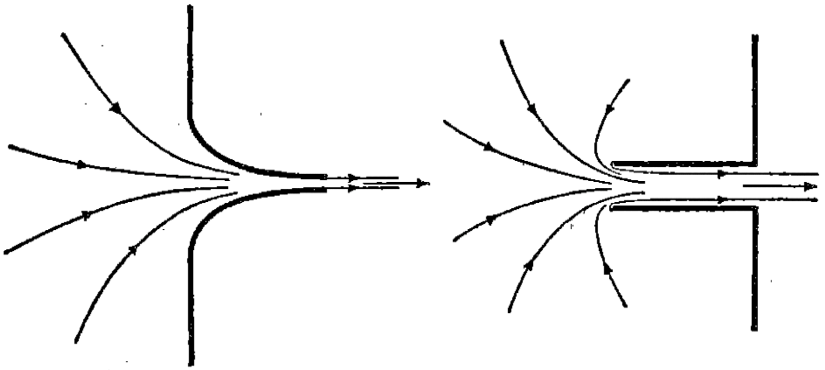
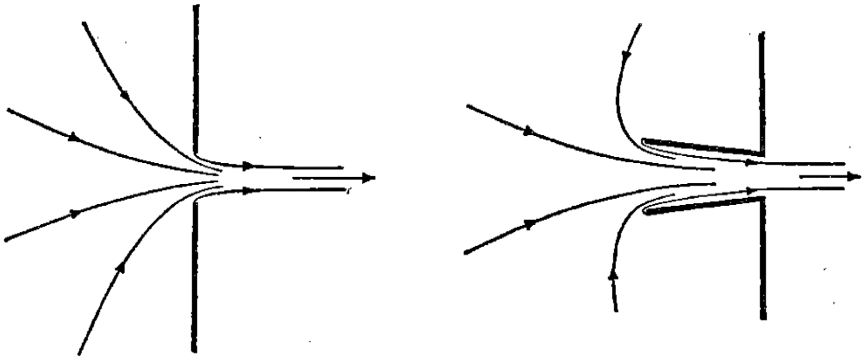
in general be determined by integral arguments of the above kind. In view of (6.3.7), a calculation of the second contribution to the reaction is effectively a calculation of the contraction coefficient α.

For two particular kinds of orifice it happens that the value of α can be obtained immediately. One is the case of a long slowly converging mouthpiece in which the streamlines become straight and parallel before emerging (figure 6.3.2(*a*)). Evidently $\alpha = 1$ here, and (6.3.7) then shows that the contribution to the reaction on the vessel from the pressure drop at the wall near the orifice is of the same magnitude as that from the uncompensated pressure on the area S of the wall opposite to the orifice. The other special case is an orifice consisting of a cylindrical tube (of internal cross-sectional area S) projecting into the vessel (figure 6.3.2(*b*)), called Borda's mouthpiece. The region in which the water velocity is appreciable is here near the entrance to the tube, and the pressure has approximately the static-fluid value $p_0 + \rho\mathbf{g}.\mathbf{x}$ at all points of the vessel wall except on the tube. No contri-



(*a*) Slowly converging mouthpiece, $\alpha = 1$

(*b*) Borda's mouthpiece, $\alpha = \frac{1}{2}$



(*c*) Hole in a plane wall, $\alpha \approx 0{\cdot}6$ (observation)

(*d*) Mouthpiece for which $\alpha < \frac{1}{2}$

Figure 6.3.2. Efflux from a circular orifice.

bution to the component of **R** in the direction of the tube axis arises from the pressure on the tube,[†] so that direct evaluation of the integral in (6.3.5) gives

$$\mathbf{k}.\mathbf{R} = -\rho g h S. \tag{6.3.8}$$

Comparison with (6.3.7) shows that $\alpha = \frac{1}{2}$ in this case.

For most other shapes of orifice α lies between $\frac{1}{2}$ and 1, since the departure of the pressure from the static-fluid value at the vessel wall near the orifice cannot be other than a suction and the rest is a matter of geometry. For the unusual orifice shown in figure 6.3.2 (d) (for which the area S is taken at the inner, or wider, end of the conical mouthpiece), the relative suction on the wetted side of the mouthpiece makes a contribution to the reaction on the vessel which is in the same direction as the jet, instead of opposite to it, so that here we see from (6.3.7) that α should be slightly less than $\frac{1}{2}$.

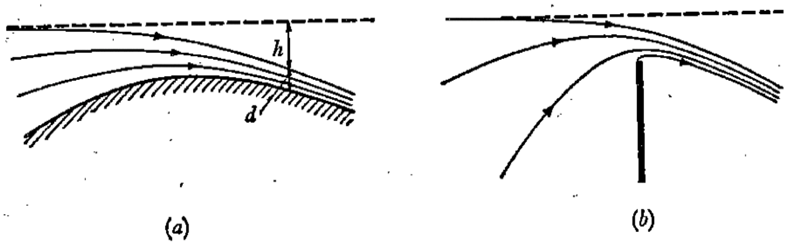


Figure 6.3.3. Steady flow over a weir, (a) broad-crested and (b) sharp-crested.

Flow over a weir

Hydraulic engineers often need to know the quantity of water flowing steadily along an open channel or through a gate of a reservoir. A simple means of obtaining this information approximately is to obstruct the flow of water, at some point in the channel or at the gate of the reservoir, by a submerged obstacle or 'weir', and to observe the level of the surface of the dammed slowly-moving water upstream of the obstacle.

A common type of weir is 'broad-crested', as sketched in figure 6.3.3 (a). The slopes of both the weir and the free water surface are here small, and we may assume that the water velocity q is approximately uniform across the stream above any point of the weir. Then if d is the depth of this stream, the rate of discharge of water volume, per unit width of the weir in a direction normal to the figure, is

$$Q = qd. \tag{6.3.9}$$

We also know from Bernoulli's theorem for the streamline at the surface that

$$\tfrac{1}{2}q^2 - gh = 0, \tag{6.3.10}$$

[†] Curiously enough, in irrotational flow round a sharp edge of exterior angle 2π, the infinitely large velocity and infinitely large suction at the edge *does* yield a non-zero force on the boundary (see § 6.5); however, here the angular extent of the region occupied by fluid near the edge is supposed to be less than 2π, owing to the separation of the jet from the sharp edge, and the force on the boundary in the direction of the tube axis is zero.

where h is the fall in water level below that some distance upstream where the velocity is negligibly small. Thus the discharge is

$$Q = \overset{q}{(2gh)^{\frac{1}{2}}}\, d, \tag{6.3.11}$$

and may be calculated from observations of both h and d at any point.

A further piece of information can be deduced by noticing that the vertical distance of any point of the weir from the level of the dammed water, viz.

$$d+h, \quad = \frac{Q}{q} + \boxed{\frac{q^2}{2g}}, \tag{6.3.12}$$

has a minimum with respect to q. Consequently, if the upstream and downstream conditions are such that the speed of a material element increases from zero in the reservoir to a value greater than $(gQ)^{\frac{1}{3}}$ as it passes over the weir, the speed $(gQ)^{\frac{1}{3}}$ occurs at the place where $d+h$ is a minimum, i.e. above the highest point of the weir.† Under these conditions, the values of h, d and q at the highest point of the weir are

$$h_1 = \tfrac{1}{2}(Q^2/g)^{\frac{1}{3}}, \quad d_1 = (Q^2/g)^{\frac{1}{3}}, \quad q_1 = (gQ)^{\frac{1}{3}}. \tag{6.3.13}$$

Thus a measurement of either h_1 or d_1 (or of h_1+d_1 more conveniently, since this quantity can be measured at a position where the water is almost still) suffices for the determination of Q.

Formulae of the kind (6.3.13) are also valid for weirs which are not broad-crested (as indeed seems likely on dimensional grounds), although the numerical coefficients are different. Weirs with a sharp edge, as shown in figure 6.3.3 (*b*), are sometimes used and here it is found by observation that

$$Q = cg^{\frac{1}{2}}(h_1+d_1)^{\frac{3}{2}}, \tag{6.3.14}$$

where c is approximately 5 % greater than the theoretical value $(\frac{2}{3})^{\frac{3}{2}}$ for a broad-crested weir. It is necessary here to allow air to have free access to the region beneath the emerging jet, for if this region is confined the air in it is gradually entrained and removed by the jet and the jet is sucked downwards.

Jet of liquid impinging on a plane wall

If a steady cylindrical jet of water surrounded by air is directed against an inclined plane rigid wall, the jet is converted into a sheet of water adjoining the wall in which the flow is everywhere away from the point of impact. We suppose that the velocity of the water in the approaching jet is uniform and of sufficiently large magnitude U for the effect of gravity to be negligible. The speed everywhere on the free surface is then equal to U, by Bernoulli's theorem. The speed within the sheet must likewise be approximately

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† The significance of the speed $(gQ)^{\frac{1}{3}}$ lies in the fact that it is also the maximum speed of propagation of surface waves of small amplitude when the depth is d_1; disturbances can be propagated into the reservoir from any point upstream of the highest point of the weir—and in this way the presence of the weir dams up the water—but not from any point downstream of it since the water speed is here greater than $(gQ)^{\frac{1}{3}}$.

uniform (except in the thin boundary layer near the wall) at some distance from the point of impact, because the velocity is there approximately unidirectional, and the pressure consequently uniform, across the sheet, again by Bernoulli's theorem. Thus all that remains to be determined about the sheet at some distance from the point of impact is the distribution of sheet thickness with respect to direction of the flow away from the point of impact. The total mass flux away from the point of impact is of course equal to that in the jet, but its directional distribution remains unknown.

Jets with circular cross-section are of special interest, since they can be produced easily in a laboratory. In laboratory work the plane rigid wall can be replaced by a plane of symmetry, that is, by directing two similar circular jets so that their two axes intersect. The resulting sheet of water is then free from the effect of viscosity at a rigid wall, and spreads out radially until its thickness (which varies as the reciprocal of the radial distance, in order to satisfy conservation of mass) is so small that it breaks up into discrete drops under the action of surface tension.

The momentum equation clearly imposes a constraint on the way in which the jet is diverted by the wall. Since there is no net force exerted by boundaries on the water in directions parallel to the wall, the momentum of the sheet is equal to the component of the jet momentum in the plane of the wall. This further relation does not enable the directional distribution of sheet thickness to be determined in general, but it is sufficient in the case of a two-dimensional jet, which produces a sheet whose thickness distribution is specified by just two values, one for each of the two streams moving away from the point of impact. We therefore proceed with the two-dimensional case, despite its limited physical importance, as a further illustration of use of the momentum equation in integral form.

Figure 6.3.4 shows the two-dimensional jet of width b impinging at angle α to the normal to the wall and dividing into two streams whose widths are ultimately uniform and equal to b_1 and b_2. We take the control surface shown as a broken line in the figure, on which the velocity is equal to U, and the pressure is equal to that in the surrounding air (p_0), except at points on the wall near the central impact point O. Then the component of the vector equation (6.3.3) parallel to the wall reduces to

$$\rho U^2(-b\sin\alpha + b_1 - b_2) = 0. \tag{6.3.15}$$

Conservation of mass of the water requires

$$b_1 + b_2 = b, \tag{6.3.16}$$

and these two relations together give

$$b_1 = \tfrac{1}{2}b(1+\sin\alpha), \quad b_2 = \tfrac{1}{2}b(1-\sin\alpha). \tag{6.3.17}$$

It is also possible to obtain some information about the distribution of pressure on the wall near O. The component of equation (6.3.3) normal to the wall becomes

$$\rho U^2 b \cos\alpha = \int (p - p_0)\, dA_w = F, \tag{6.3.18}$$

where the integral is taken over the surface of the wall, and F is the magnitude of the normal force exerted on the wall by the jet (both per unit depth, normal to the plane of the figure). Furthermore, consideration of the moment about O of the momentum entering and leaving this same control region and of the forces acting on the fluid within the region shows that the centre of pressure C (that is, the point at which a concentrated normal force F on the wall has an anti-clockwise moment about O equal to that of the pressure distribution on the wall) is a point on the anti-clockwise side of the normal and at a distance from O given by

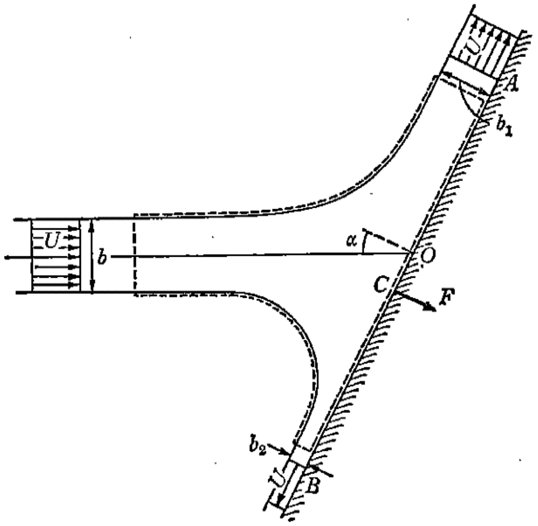

Figure 6.3.4. Jet of liquid impinging on an inclined plane wall (two-dimensional case).

$$OC \times F = \tfrac{1}{2}\rho U^2 b_1^2 - \tfrac{1}{2}\rho U^2 b_2^2.$$

Hence
$$OC = \tfrac{1}{2} b \tan\alpha. \tag{6.3.19}$$

Thus if the rigid wall were hinged about O it would tend to set itself at right angles to the jet.

This latter qualitative result holds also for a rigid flat plate of finite breadth hinged about one of the bisecting lines and immersed in a stream of great width; and rectangular plates falling through infinite fluid tend to fall broadside-on. The explanation is to be found in the location of the stagnation point, where the pressure is a maximum, on the wall or plate. For two reasons the stagnation point moves away from the central point O, as α increases from zero, towards the point B in figure 6.3.4. The first is that more of the fluid in the oncoming jet flows towards the point A, so that the dividing streamline